MATRIX METHODS OF ANALYSIS

INTRODUCTION

The mathematical language which is most convenient for analyzing multiple degree-of-freedom vibratory systems is that of *matrices*. Matrix notation simplifies the preliminary analytical study, and in situations where particular numerical answers are required, matrices provide a standardized format for organizing the data and the computations. Computations with matrices can be carried out by hand or by digital computers. The availability of programs such as MATLAB makes the solution of many complex problems in vibration analysis a matter of routine.

This chapter describes how matrices are used in vibration analysis. It begins with definitions and rules for operating with matrices. The formulation of vibration problems in matrix notation then is treated. This is followed by general matrix solutions of several important types of vibration problems, including free and forced vibrations of both undamped and damped linear multiple degree-of-freedom systems. Part II of this chapter considers finite element models.

MATRICES

Matrices are mathematical entities which facilitate the handling of simultaneous equations. They are applied to the differential equations of a vibratory system as follows:

A single degree-of-freedom system of the type in Fig. 28.1 has the differential equation

$$m\ddot{x} + c\dot{x} + kx = F$$

where m is the mass, c is the damping coefficient, k is the stiffness, F is the applied force, x is the displacement coordinate, and dots denote time derivatives. In Fig. 28.2 a similar three degree-of-freedom system is shown. The equations of motion may be obtained by applying Newton's second law to each mass in turn:

$$\begin{aligned} m\ddot{x}_1 \quad &+ c\dot{x}_1 \quad &&+ 5kx_1 - 2kx_2 \quad &&= F_1 \\ 2m\ddot{x}_2 \quad &&&+ 2c\dot{x}_2 - 2c\dot{x}_3 - 2kx_1 + 3kx_2 - kx_3 = F_2 \\ 3m\ddot{x}_3 \quad &&&- 2c\dot{x}_2 + 2c\dot{x}_3 \quad &&- kx_2 + kx_3 = F_3 \end{aligned} \tag{28.1}$$

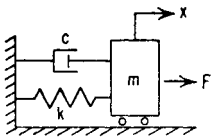

FIGURE 28.1 Single degree-of-freedom system.

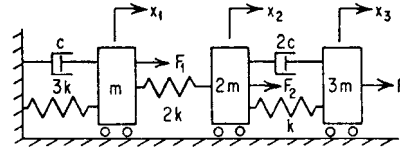

FIGURE 28.2 Three degree-of-freedom system.

The accelerations, velocities, displacements, and forces may be organized into columns, denoted by single boldface symbols:

$$\ddot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} \qquad \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{28.2}$$

The inertia, damping, and stiffness coefficients may be organized into square arrays:

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 3m \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} c & 0 & 0 \\ 0 & 2c & -2c \\ 0 & -2c & 2c \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 5k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} \tag{28.3}$$

By using these symbols, it is shown below that it is possible to represent the three equations of Eq. (28.1) by the following single equation:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{28.4}$$

Note that this has the same form as the differential equation for the single degree-of-freedom system of Fig. 28.1. The notation of Eq. (28.4) has the advantage that in systems of many degrees-of-freedom it clearly states the physical principle that at every coordinate the external force is the sum of the inertia, damping, and stiffness forces. Equation (28.4) is an abbreviation for Eq. (28.1). It is necessary to develop the rules of operation with symbols such as those in Eqs. (28.2) and (28.3) to ensure that no ambiguity is involved. The algebra of *matrices* is devised to facilitate manipulations of simultaneous equations such as Eq. (28.1). Matrix algebra does not in any way simplify individual operations such as multiplication or addition of numbers, but it is an organizational tool which permits one to keep track of a complicated sequence of operations in an optimum manner. Matrices are essential elements of linear algebra,[1] and are widely employed in structural analysis[2] and vibration analysis.[3]

DEFINITIONS

A *matrix* is an array of elements arranged systematically in rows and columns. For example, a rectangular matrix $\mathbf{A}$, of elements a_{jk}, which has m rows and n columns is

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The elements a_{jk} are usually numbers or functions, but, in principle, they may be any well-defined quantities. The first subscript j on the element refers to the row number while the second subscript k refers to the column number. The array is denoted by the single symbol **A**, which can be used as such during operational manipulations in which it is not necessary to specify continually all the elements a_{jk}. When a numerical calculation is finally required, it is necessary to refer back to the explicit specifications of the elements a_{jk}.

A rectangular matrix with m rows and n columns is said to be of order (m,n). A matrix of order (n,n) is a *square matrix* and is said to be simply a square matrix of order n. A matrix of order $(n,1)$ is a *column matrix* and is said to be simply a column matrix of order n. A column matrix is sometimes referred to as a *column vector.* Similarly, a matrix of order $(1,n)$ is a *row matrix* or a *row vector.* Boldface *capital* letters are used here to represent square matrices and *lower-case* boldface letters to represent column matrices or vectors. For example, the matrices in Eq. (28.2) are column matrices of order three and the matrices in Eq. (28.3) are square matrices of order three.

Some special types of matrices are:

1. A *diagonal matrix* is a square matrix **A** whose elements a_{jk} are zero when $j \neq k$. The only nonzero elements are those on the *main diagonal,* where $j = k$. In order to emphasize that a matrix is diagonal, it is often written with small ticks in the direction of the main diagonal:

$$\mathbf{A} = \begin{bmatrix} a_{jj} \end{bmatrix}$$

2. A *unit matrix* or *identity matrix* is a diagonal matrix whose main diagonal elements are each equal to unity. The symbol **I** is used to denote a unit matrix. Examples are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. A *null matrix* or *zero matrix* has all its elements equal to zero and is simply written as zero.

4. The *transpose* $\mathbf{A}^T$ of a matrix **A** is a matrix having the same elements but with rows and columns interchanged. Thus, if the original matrix is

$$\mathbf{A} = [a_{jk}]$$

the transpose matrix is

$$\mathbf{A}^T = [a_{jk}]^T = [a_{kj}]$$

For example:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$$

The transpose of a square matrix may be visualized as the matrix obtained by rotating the given matrix about its main diagonal as an axis.

The transpose of a column matrix is a row matrix. For example,

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} \qquad \mathbf{x}^T = [3 \quad 4 \quad -2]$$

Throughout this chapter a row matrix is referred to as the transpose of the corresponding column matrix.

5. A *symmetric matrix* is a square matrix whose off-diagonal elements are symmetric with respect to the main diagonal. A square matrix **A** is symmetric if, for all j and k,

$$a_{jk} = a_{kj}$$

A symmetric matrix is equal to its transpose. For example, all three of the matrices in Eq. (28.3) are symmetric. In addition, the matrix **M** is a diagonal matrix.

MATRIX OPERATIONS

Equality of Matrices. Two matrices of the same order are equal if their corresponding elements are equal. Thus two matrices **A** and **B** are equal if, for every j and k,

$$a_{jk} = b_{jk}$$

Matrix Addition and Subtraction. Addition or subtraction of matrices of the same order is performed by adding or subtracting corresponding elements. Thus, $\mathbf{A} + \mathbf{B} = \mathbf{C}$ if for every j and k,

$$a_{jk} + b_{jk} = c_{jk}$$

For example, if

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 5 & 6 \end{bmatrix}$$

then

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix} \qquad \mathbf{A} - \mathbf{B} = \begin{bmatrix} 4 & 0 \\ -6 & -2 \end{bmatrix}$$

Multiplication of a Matrix by a Scalar. Multiplication of a matrix by a scalar c multiplies each element of the matrix by c. Thus

$$c\mathbf{A} = c[a_{jk}] = [ca_{jk}]$$

In particular, the negative of a matrix has the sign of every element changed.

Matrix Multiplication. If **A** is a matrix of order (m,n) and **B** is a matrix of order (n,p), then their *matrix product* $\mathbf{AB} = \mathbf{C}$ is defined to be a matrix **C** of order (m,p) where, for every j and k,

$$c_{jk} = \sum_{r=1}^{n} a_{jr} b_{rk} \tag{28.5}$$

The product of two matrices can be obtained only if they are *conformable,* i.e., if the number of columns in **A** is equal to the number of rows in **B**. The symbolic equation

$$(m,n) \times (n,p) = (m,p)$$

indicates the orders of the matrices involved in a matrix product. Matrix products are not commutative, i.e., in general,

$$\mathbf{AB} \neq \mathbf{BA}$$

The matrix products which appear in this chapter are of the following types:

Square matrix × square matrix = square matrix
Square matrix × column vector = column vector
Row vector × square matrix = row vector
Row vector × column vector = scalar
Column vector × row vector = square matrix

In all cases, the matrices must be conformable. Numerical examples are given below.

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -(3\times 1) + (2\times 5) & (3\times 2) + (2\times 6) \\ (1\times 1) + (4\times 5) & -(1\times 2) + (4\times 6) \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 21 & 22 \end{bmatrix}$$

$$\mathbf{Ax} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} (3\times 5) + (2\times 3) \\ -(1\times 5) + (4\times 3) \end{bmatrix} = \begin{bmatrix} 21 \\ 7 \end{bmatrix}$$

$$\mathbf{y}^T\mathbf{A} = [-2 \quad 1] \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} = [-(2\times 3) - (1\times 1) - (2\times 2) + (1\times 4)] = [-7 \quad 0]$$

$$\mathbf{y}^T\mathbf{x} = [-2 \quad 1] \begin{bmatrix} 5 \\ 3 \end{bmatrix} = (-10 + 3) = -7$$

$$\mathbf{x}\mathbf{y}^T = \begin{bmatrix} 5 \\ 3 \end{bmatrix} [-2 \quad 1] = \begin{bmatrix} -(5\times 2) & (5\times 1) \\ -(3\times 2) & (3\times 1) \end{bmatrix} = \begin{bmatrix} -10 & 5 \\ -6 & 3 \end{bmatrix}$$

The last product always results in a matrix with proportional rows and columns.

The operation of matrix multiplication is particularly suited for representing systems of simultaneous linear equations in a compact form in which the coefficients are gathered into square matrices and the unknowns are placed in column matrices. For example, it is the operation of matrix multiplication which gives unambiguous meaning to the matrix abbreviation in Eq. (28.4) for the three simultaneous differential equations of Eq. (28.1). The two sides of Eq. (28.4) are column matrices of order three whose corresponding elements must be equal. On the right, these elements are simply the external forces at the three masses. On the left, Eq. (28.4) states that the resulting column is the sum of three column matrices, each of which results from the matrix multiplication of a square matrix of coefficients defined in Eq. (28.3) into a column matrix defined in Eq. (28.2). The rules of matrix operation just given ensure that Eq. (28.4) is exactly equivalent to Eq. (28.1).

Premultiplication or postmultiplication of a square matrix by the identity matrix leaves the original matrix unchanged; i.e.,

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$

Two symmetrical matrices multiplied together are generally not symmetric. The product of a matrix and its transpose is symmetric.

Continued matrix products such as **ABC** are defined, provided the number of columns in each matrix is the same as the number of rows in the matrix immediately following it. From the definition of matrix products, it follows that the *associative law* holds for continued products:

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

A square matrix **A** multiplied by itself yields a square matrix which is called the *square of the matrix* **A** and is denoted by $\mathbf{A}^2$. If $\mathbf{A}^2$ is in turn multiplied by **A**, the resulting matrix is $\mathbf{A}^3 = \mathbf{A}(\mathbf{A}^2) = \mathbf{A}^2(\mathbf{A})$. Extension of this process gives meaning to $\mathbf{A}^m$ for any positive integer *power m.* Powers of symmetric matrices are themselves symmetric.

The rule for *transposition* of matrix products is

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

Inverse or Reciprocal Matrix. If, for a given square matrix **A**, a square matrix $\mathbf{A}^{-1}$ can be found such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{28.6}$$

then $\mathbf{A}^{-1}$ is called the *inverse* or *reciprocal* of **A**. Not every square matrix **A** possesses an inverse. If the determinant constructed from the elements of a square matrix is zero, the matrix is said to be *singular* and there is no inverse. Every nonsingular matrix possesses a unique inverse. The inverse of a symmetric matrix is symmetric. The rule for the *inverse of a matrix product* is

$$(\mathbf{AB})^{-1} = (\mathbf{B}^{-1})(\mathbf{A}^{-1})$$

The solution to the set of simultaneous equations

$$\mathbf{Ax} = \mathbf{c}$$

where **x** is the unknown vector and **c** is a known input vector can be indicated with the aid of the inverse of **A**. The formal solution for **x** proceeds as follows:

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{c}$$

$$\mathbf{Ix} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{c}$$

When the inverse $\mathbf{A}^{-1}$ is known, the solution vector **x** is obtained by a simple matrix multiplication of $\mathbf{A}^{-1}$ into the input vector **c**.

Calculation of inverses and the solutions of simultaneous linear equations are readily performed for surprisingly large values of n by programs such as MATLAB. When $n = 2$ and

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

hand-computation is possible using the following formulas:

$$\mathbf{A}^{-1} = \frac{1}{\Delta}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \qquad x_1 = \frac{\Delta_1}{\Delta} \qquad x_2 = \frac{\Delta_2}{\Delta}$$

where the determinants have the values

$$\Delta = a_{11}a_{22} - a_{12}a_{21} \qquad \Delta_1 = c_1a_{22} - c_2a_{12} \qquad \Delta_2 = c_2a_{11} - c_1a_{21}$$

QUADRATIC FORMS

A general quadratic form Q of order n may be written as

$$Q = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk}x_jx_k$$

where the a_{jk} are constants and the x_j are the n variables. The form is quadratic since it is of the second degree in the variables. The laws of matrix multiplication permit Q to be written as

$$Q = [x_1 x_2 \ldots x_n] \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$$

which is

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

Any quadratic form can be expressed in terms of a symmetric matrix. If the given matrix **A** is not symmetric, it can be replaced by the symmetric matrix

$$\mathbf{B} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

without changing the value of the form.

As an example of a quadratic form, the *potential energy* V for the system of Fig. 28.2 is given by

$$\begin{aligned} 2V &= 3kx_1^2 \quad + 2k(x_2 - x_1)^2 + k(x_3 - x_2)^2 \\ &= 5kx_1x_1 - 2kx_1x_2 \\ &\quad - 2kx_2x_1 + 3kx_2x_2 - kx_2x_3 \\ &\quad - kx_3x_2 + kx_3x_3 \end{aligned}$$

Using the displacement vector **x** defined in Eq. (28.2) and the stiffness matrix **K** in Eq. (28.3), the potential energy may be written as

$$V = \tfrac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x}$$

Similarly, the *kinetic energy* T is given by

$$2T = m\dot{x}_1^2 + 2m\dot{x}_2^2 + 3m\dot{x}_3^2$$

In terms of the inertia matrix **M** and the velocity vector $\dot{\mathbf{x}}$ defined in Eqs. (28.3) and (28.2), the kinetic energy may be written as

$$T = \tfrac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}}$$

The *dissipation function* D for the system is given by

$$\begin{aligned} 2D &= c\dot{x}_1^2 + 2c(\dot{x}_3 - \dot{x}_2)^2 \\ &= c\dot{x}_1\dot{x}_1 \\ &\quad + 2c\dot{x}_2\dot{x}_2 - 2c\dot{x}_2\dot{x}_3 \\ &\quad - 2c\dot{x}_3\dot{x}_2 + 2c\dot{x}_3\dot{x}_3 \end{aligned}$$

In terms of the velocity vector $\dot{\mathbf{x}}$ and the damping matrix $\mathbf{C}$ defined in Eqs. (28.2) and (28.3), the dissipation function may be written as

$$D = \tfrac{1}{2}\dot{\mathbf{x}}^T\mathbf{C}\dot{\mathbf{x}}$$

The dissipation function gives half the rate at which energy is being dissipated in the system.

While quadratic forms assume positive and negative values in general, the three physical forms just defined are intrinsically *positive* for a vibrating system with linear springs, constant masses, and viscous damping; i.e., they can never be negative for a real motion of the system. Kinetic energy is zero only when the system is at rest. The same thing is not necessarily true for potential energy or the dissipation function.

Depending upon the arrangement of springs and dashpots in the system, there may exist motions which do not involve any potential energy or dissipation. For example, in vibratory systems where rigid body motions are possible (crankshaft torsional systems, free-free beams, etc.), no elastic energy is involved in the rigid body motions. Also, in Fig. 28.2, if x_1 is zero while x_2 and x_3 have the same motion, there is no energy dissipated and the dissipation function is zero. To distinguish between these two possibilities, a quadratic form is called *positive definite* if it is never negative and if the only time it vanishes is when all the variables are zero. Kinetic energy is always positive definite, while potential energy and the dissipation function are positive but not necessarily positive definite. It depends upon the particular configuration of a given system whether the potential energy and the dissipation function are positive definite or only positive. The terms positive and positive definite are applied also to the matrices from which the quadratic forms are derived. For example, of the three matrices defined in Eq. (28.3), the matrices $\mathbf{M}$ and $\mathbf{K}$ are positive definite, but $\mathbf{C}$ is only positive. It can be shown that a matrix which is positive but not positive definite is *singular.*

Differentiation of Quadratic Forms. In forming Lagrange's equations of motion for a vibrating system,* it is necessary to take derivatives of the potential energy V, the kinetic energy T, and the dissipation function D. When these quadratic forms are represented in matrix notation, it is convenient to have matrix formulas for differentiation. In this paragraph rules are given for differentiating the slightly more general *bilinear form*

$$F = \mathbf{x}^T\mathbf{A}\mathbf{y} = \mathbf{y}^T\mathbf{A}\mathbf{x}$$

where $\mathbf{x}^T$ is a row vector of n variables x_j, $\mathbf{A}$ is a square matrix of constant coefficients, and $\mathbf{y}$ is a column matrix of n variables y_j. In a quadratic form the x_j are identical with the y_j.

For generality it is assumed that the x_j and the y_j are functions of n other variables u_j. In the formulas below, the notation $\mathbf{X}_u$ is used to represent the following square matrix:

* See Chap. 2 for a detailed discussion of Lagrange's equations.

$$\mathbf{X}_u = \begin{bmatrix} \dfrac{\partial x_1}{\partial u_1} & \dfrac{\partial x_2}{\partial u_1} & \cdots & \dfrac{\partial x_n}{\partial u_1} \\ \dfrac{\partial x_1}{\partial u_2} & \dfrac{\partial x_2}{\partial u_2} & \cdots & \dfrac{\partial x_n}{\partial u_2} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial u_n} & \dfrac{\partial x_2}{\partial u_n} & \cdots & \dfrac{\partial x_n}{\partial u_n} \end{bmatrix}$$

Now letting $\partial/\partial\mathbf{u}$ stand for the column vector whose elements are the partial differential operators with respect to the u_j, the general differentiation formula is

$$\frac{\partial F}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial F}{\partial u_1} \\ \dfrac{\partial F}{\partial u_2} \\ \vdots \\ \dfrac{\partial F}{\partial u_n} \end{bmatrix} = \mathbf{X}_u\mathbf{A}\mathbf{y} + \mathbf{Y}_u\mathbf{A}^T\mathbf{x}$$

For a quadratic form $Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$ the above formula reduces to

$$\frac{\partial Q}{\partial \mathbf{u}} = \mathbf{X}_u(\mathbf{A} + \mathbf{A}^T)\mathbf{x}$$

Thus whether $\mathbf{A}$ is symmetric or not, this kind of differentiation produces a *symmetrical* matrix of coefficients $(\mathbf{A} + \mathbf{A}^T)$. It is this fact which ensures that vibration equations in the form obtained from Lagrange's equations always have symmetrical matrices of coefficients. If $\mathbf{A}$ is symmetrical to begin with, the previous formula becomes

$$\frac{\partial Q}{\partial \mathbf{u}} = 2\mathbf{X}_u\mathbf{A}\mathbf{x}$$

Finally, in the important special case where the x_j are identical with the u_j, the matrix $\mathbf{X}_x$ reduces to the identity matrix, yielding

$$\frac{\partial Q}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x} \tag{28.7}$$

which is employed in the following section in developing Lagrange's equations.

FORMULATION OF VIBRATION PROBLEMS IN MATRIX FORM

Consider a holonomic linear mechanical system with n degrees-of-freedom which vibrates about a stable equilibrium configuration. Let the motion of the system be described by *n generalized displacements* $x_j(t)$ which vanish in the equilibrium position. The potential energy V can then be expressed in terms of these displacements as a quadratic form. The kinetic energy T and the dissipation function D can be expressed as quadratic forms in the generalized velocities $\dot{x}_j(t)$.

The equations of motion are obtained by applying Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) + \frac{\partial D}{\partial \dot{x}_j} + \frac{\partial V}{\partial x_j} = f_j(t) \qquad [j = 1, 2, \ldots, n]$$

The *generalized external force* $f_j(t)$ for each coordinate may be an active force in the usual sense or a force generated by prescribed motion of the coordinates.

If each term in the foregoing equation is taken as the jth element of a column matrix, all n equations can be considered simultaneously and written in matrix form as follows:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{x}}}\right) + \frac{\partial D}{\partial \dot{\mathbf{x}}} + \frac{\partial V}{\partial \mathbf{x}} = \mathbf{f}$$

The quadratic forms can be expressed in matrix notation as

$$T = \tfrac{1}{2}(\dot{\mathbf{x}}^T \mathbf{M} \dot{\mathbf{x}})$$

$$D = \tfrac{1}{2}(\dot{\mathbf{x}}^T \mathbf{C} \dot{\mathbf{x}})$$

$$V = \tfrac{1}{2}(\mathbf{x}^T \mathbf{K} \mathbf{x})$$

where the *inertia matrix* $\mathbf{M}$, the *damping matrix* $\mathbf{C}$, and the *stiffness matrix* $\mathbf{K}$ may be taken as symmetric square matrices of order n. Then the differentiation rule (28.7) yields

$$\frac{d}{dt}(\mathbf{M}\dot{\mathbf{x}}) + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}$$

or simply

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{28.8}$$

as the equations of motion in matrix form for a general linear vibratory system with n degrees-of-freedom. This is a generalization of Eq. (28.4) for the three degree-of-freedom system of Fig. 28.2. Equation (28.8) applies to all linear constant-parameter vibratory systems. The specifications of any particular system are contained in the *coefficient matrices* $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$. The type of excitation is described by the column matrix $\mathbf{f}$. The individual terms in the coefficient matrices have the following significance:

m_{jk} is the momentum component at j due to a unit velocity at k.

c_{jk} is the damping force at j due to a unit velocity at k.

k_{jk} is the elastic force at j due to a unit displacement at k.

The general solution to Eq. (28.8) contains $2n$ constants of integration which are usually fixed by the n displacements $x_j(t_0)$ and the n velocities $\dot{x}_j(t_0)$ at some initial time t_0. When the excitation matrix $\mathbf{f}$ is zero, Eq. (28.8) is said to describe the *free vibration* of the system. When $\mathbf{f}$ is nonzero, Eq. (28.8) describes a *forced vibration*. When the time behavior of $\mathbf{f}$ is periodic and steady, it is sometimes convenient to divide the solution into a *steady-state response* plus a *transient response* which decays with time. The steady-state response is independent of the initial conditions.

COUPLING OF THE EQUATIONS

The off-diagonal terms in the coefficient matrices are known as *coupling terms.* In general, the equations have inertia, damping, and stiffness coupling; however, it is often possible to obtain equations that have no coupling terms in one or more of the three matrices. If the coupling terms vanish in all three matrices (i.e., if all three square matrices are diagonal matrices), the system of Eq. (28.8) becomes a set of independent uncoupled differential equations for the n generalized displacements $x_j(t)$. Each displacement motion is a single degree-of-freedom vibration independent of the motion of the other displacements.

The coupling in a system depends on the choice of coordinates used to describe the motion. For example, Figs. 28.3 and 28.4 show the same physical system with two different choices for the displacement coordinates.

The coefficient matrices corresponding to the coordinates shown in Fig. 28.3 are

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Here the inertia matrix is uncoupled because the coordinates chosen are the absolute displacements of the masses. The elastic force in the spring k_2 is generated by the relative displacement of the two coordinates, which accounts for the coupling terms in the stiffness matrix.

The coefficient matrices corresponding to the alternative coordinates shown in Fig. 28.4 are

$$\mathbf{M} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

Here the coordinates chosen relate directly to the extensions of the springs so that the stiffness matrix is uncoupled. The absolute displacement of m_2 is, however, the sum of the coordinates, which accounts for the coupling terms in the inertia matrix.

A fundamental procedure for solving vibration problems in undamped systems may be viewed as the search for a set of coordinates which simultaneously uncouples both the stiffness and inertia matrices. This is always possible. In systems with damping (i.e., with all three coefficient matrices) there exist coordinates which uncouple two of these, but it is not possible to uncouple all three matrices simultaneously, except in the special case, called *proportional damping,* where **C** is a linear combination of **K** and **M**.

The system of Fig. 28.2 provides an example of a three degree-of-freedom system with damping. The coefficient matrices are given in Eq. (28.3). The inertia matrix is uncoupled, but the damping and stiffness matrices are coupled.

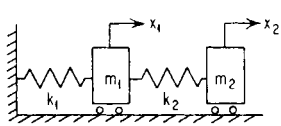

FIGURE 28.3 Coordinates (x_1,x_2) with uncoupled inertia matrix.

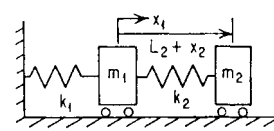

FIGURE 28.4 Coordinates (x_1,x_2) with uncoupled stiffness matrix. The equilibrium length of the spring k_2 is L_2.

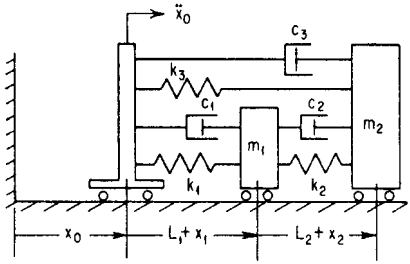

FIGURE 28.5 Two degree-of-freedom vibratory system. The equilibrium length of the spring k_1 is L_1 and the equilibrium length of the spring k_2 is L_2.

Another example of a system with damping is furnished by the two degree-of-freedom system shown in Fig. 28.5. The excitation here is furnished by acceleration $\ddot{x}_0(t)$ of the base. This system is used as the basis for the numerical example at the end of Part I of the chapter. With the coordinates chosen as indicated in the figure, all three coefficient matrices have coupling terms. The equations of motion can be placed in the standard form of Eq. (28.8), where the coefficient matrices and the excitation column are as follows:

$$\mathbf{M} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} c_1 + c_3 & c_3 \\ c_3 & c_2 + c_3 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_3 & k_3 \\ k_3 & k_2 + k_3 \end{bmatrix} \qquad \mathbf{f} = -\ddot{x}_0 \begin{bmatrix} m_1 + m_2 \\ m_2 \end{bmatrix} \tag{28.9}$$

THE MATRIX EIGENVALUE PROBLEM

In the following sections the solutions to both free and forced vibration problems are given in terms of solutions to a specialized algebraic problem known as the matrix eigenvalue problem. In the present section a general theoretical discussion of the matrix eigenvalue problem is given.

The free vibration equation for an undamped system,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0 \tag{28.10}$$

follows from Eq. (28.8) when the excitation **f** and the damping **C** vanish. If a solution for **x** is assumed in the form

$$\mathbf{x} = \Re\{\mathbf{v}e^{j\omega t}\}$$

where **v** is a column vector of unknown amplitudes, ω is an unknown frequency, j is the square root of -1, and $\Re\{\ \}$ signifies "the real part of," it is found on substituting in Eq. (28.10) that it is necessary for **v** and ω to satisfy the following algebraic equation:

$$\mathbf{K}\mathbf{v} = \omega^2\mathbf{M}\mathbf{v} \tag{28.11}$$

This algebraic problem is called the *matrix eigenvalue problem.* Where necessary it is called the *real* eigenvalue problem to distinguish it from the *complex* eigenvalue problem described in the section on *Vibration of Systems with Damping.*

To indicate the formal solution to Eq. (28.11), it is rewritten as

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{v} = 0 \tag{28.12}$$

which can be interpreted as a set of n homogeneous algebraic equations for the n elements v_j. This set always has the trivial solution

$$\mathbf{v} = 0$$

It also has nontrivial solutions if the determinant of the matrix multiplying the vector v is zero, i.e., if

$$\det (\mathbf{K} - \omega^2\mathbf{M}) = 0 \tag{28.13}$$

When the determinant is expanded, a polynomial of order n in ω^2 is obtained. Equation (28.13) is known as the *characteristic equation* or *frequency equation.* The restrictions that **M** and **K** be symmetric and that **M** be positive definite are sufficient to ensure that there are n real roots for ω^2. If **K** is singular, at least one root is zero. If **K** is positive definite, all roots are positive. The n roots determine the *n natural frequencies* ω_r $(r = 1, \ldots, n)$ of free vibration. These roots of the characteristic equation are also known as *normal values, characteristic values, proper values, latent roots,* or *eigenvalues.* When a natural frequency ω_r is known, it is possible to return to Eq. (28.12) and solve for the corresponding vector $\mathbf{v}_r$ to within a multiplicative constant. The eigenvalue problem does not fix the absolute amplitude of the vectors **v**, only the relative amplitudes of the n coordinates. There are n independent vectors $\mathbf{v}_r$ corresponding to the n natural frequencies which are known as *natural modes.* These vectors are also known as *normal modes, characteristic vectors, proper vectors, latent vectors,* or *eigenvectors.*

MODAL AND SPECTRAL MATRICES

The complete solution to the eigenvalue problem of Eq. (28.11) consists of n eigenvalues and n corresponding eigenvectors. These can be assembled compactly into matrices. Let the eigenvector $\mathbf{v}_r$ corresponding to the eigenvalue ω_r^2 have elements v_{jr} (the first subscript indicates which row, the second subscript indicates which eigenvector). The n eigenvectors then can be displayed in a single square matrix **V**, each column of which is an eigenvector:

$$\mathbf{V} = [v_{jk}] = \begin{bmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ v_{n1} & v_{n2} & \ldots & v_{nn} \end{bmatrix}$$

The matrix **V** is called the *modal matrix* for the eigenvalue problem, Eq. (28.11).

The n eigenvalues ω_r^2 can be assembled into a diagonal matrix $\mathbf{\Omega}^2$ which is known as the *spectral matrix* of the eigenvalue problem, Eq. (28.11)

$$\mathbf{\Omega}^2 = [\omega_r^2] = \begin{bmatrix} \omega_1^2 & 0 & \ldots & 0 \\ 0 & \omega_2^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \omega_n^2 \end{bmatrix}$$

Each eigenvector and corresponding eigenvalue satisfy a relation of the following form:

$$\mathbf{K}\mathbf{v}_r = \mathbf{M}\mathbf{v}_r\omega_r^2$$

By using the modal and spectral matrices it is possible to assemble all of these relations into a single matrix equation

$$\mathbf{KV} = \mathbf{MV}\boldsymbol{\Omega}^2 \tag{28.14}$$

Equation (28.14) provides a compact display of the complete solution to the eigenvalue problem Eq. (28.11).

PROPERTIES OF THE SOLUTION

The eigenvectors corresponding to different eigenvalues can be shown to satisfy the following *orthogonality relations.* When $\omega_r^2 \neq \omega_s^2$,

$$\mathbf{v}_r^T\mathbf{K}\mathbf{v}_s = 0 \qquad \mathbf{v}_r^T\mathbf{M}\mathbf{v}_s = 0 \tag{28.15}$$

In case the characteristic equation has a p-fold multiple root for ω^2, then there is a p-fold infinity of corresponding eigenvectors. In this case, however, it is always possible to choose p of these vectors which mutually satisfy Eq. (28.15) and to express any other eigenvector corresponding to the multiple root as a linear combination of the p vectors selected. If these p vectors are included with the eigenvectors corresponding to the other eigenvalues, a set of n vectors is obtained which satisfies the orthogonality relations of Eq. (28.15) for any $r \neq s$.

The orthogonality of the eigenvectors with respect to **K** and **M** implies that the following square matrices are *diagonal.*

$$\begin{aligned} \mathbf{V}^T\mathbf{K}\mathbf{V} &= \lceil \mathbf{v}_r^T\mathbf{K}\mathbf{v}_r \rfloor \\ \mathbf{V}^T\mathbf{M}\mathbf{V} &= \lceil \mathbf{v}_r^T\mathbf{M}\mathbf{v}_r \rfloor \end{aligned} \tag{28.16}$$

The elements $\mathbf{v}_r^T\mathbf{K}\mathbf{v}_r$ along the main diagonal of $\mathbf{V}^T\mathbf{K}\mathbf{V}$ are called the *modal stiffnesses* k_r, and the elements $\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r$ along the main diagonal of $\mathbf{V}^T\mathbf{M}\mathbf{V}$ are called the *modal masses* m_r. Since **M** is positive definite, all modal masses are guaranteed to be positive. When **K** is singular, at least one of the modal stiffnesses will be zero. Each eigenvalue ω_r^2 is the quotient of the corresponding modal stiffness divided by the corresponding modal mass; i.e.,

$$\omega_r^2 = \frac{k_r}{m_r}$$

In numerical work it is sometimes convenient to normalize each eigenvector so that its largest element is *unity.* In other applications it is common to normalize the eigenvectors so that the modal masses m_r all have the *same* value m, where m is some convenient value such as the total mass of the system. In this case,

$$\mathbf{V}^T\mathbf{M}\mathbf{V} = m\mathbf{I} \tag{28.17}$$

and it is possible to express the inverse of the modal matrix **V** simply as

$$\mathbf{V}^{-1} = \frac{1}{m}\,\mathbf{V}^T\mathbf{M}$$

An interpretation of the modal matrix **V** can be given by showing that it defines a set of generalized coordinates for which both the inertia and stiffness matrices are uncoupled. Let $\mathbf{y}(t)$ be a column of displacements related to the original displacements $\mathbf{x}(t)$ by the following simultaneous equations:

$$\mathbf{y} = \mathbf{V}^{-1}\mathbf{x} \qquad \text{or} \qquad \mathbf{x} = \mathbf{V}\mathbf{y}$$

The potential and kinetic energies then take the forms

$$V = \tfrac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} = \tfrac{1}{2}\mathbf{y}^T(\mathbf{V}^T\mathbf{K}\mathbf{V})\mathbf{y}$$

$$T = \tfrac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} = \tfrac{1}{2}\dot{\mathbf{y}}^T(\mathbf{V}^T\mathbf{M}\mathbf{V})\dot{\mathbf{y}}$$

where, according to Eq. (28.16), the square matrices in parentheses on the right are *diagonal;* i.e., in the y_j coordinate system there is neither stiffness nor inertia coupling.

An alternative method for obtaining the same interpretation is to start from the eigenvalue problem of Eq. (28.11). Consider the structure of the related eigenvalue problem for **w** where again **w** is obtained from **v** by the transformation involving the modal matrix **V**.

$$\mathbf{w} = \mathbf{V}^{-1}\mathbf{v} \qquad \text{or} \qquad \mathbf{v} = \mathbf{V}\mathbf{w}$$

Substituting in Eq. (28.11), premultiplying by $\mathbf{V}^T$, and using Eq. (28.14),

$$\mathbf{K}\mathbf{v} = \omega^2\mathbf{M}\mathbf{v}$$

$$\mathbf{K}\mathbf{V}\mathbf{w} = \omega^2\mathbf{M}\mathbf{V}\mathbf{w}$$

$$\mathbf{V}^T\mathbf{K}\mathbf{V}\mathbf{w} = \omega^2\mathbf{V}^T\mathbf{M}\mathbf{V}\mathbf{w}$$

$$(\mathbf{V}^T\mathbf{M}\mathbf{V})\boldsymbol{\Omega}^2\mathbf{w} = \omega^2(\mathbf{V}^T\mathbf{M}\mathbf{V})\mathbf{w}$$

Now, since $\mathbf{V}^T\mathbf{M}\mathbf{V}$ is a diagonal matrix of positive elements, it is permissible to cancel it from both sides, which leaves a simple diagonalized eigenvalue problem for **w**:

$$\boldsymbol{\Omega}^2\mathbf{w} = \omega^2\mathbf{w}$$

A modal matrix for **w** is the identity matrix **I**, and the eigenvalues for **w** are the same as those for **v**.

EIGENVECTOR EXPANSIONS

Any set of n independent vectors can be used as a basis for representing any other vector of order n. In the following sections, the eigenvectors of the eigenvalue problem of Eq. (28.11) are used as such a basis. An eigenvector expansion of an arbitrary vector **y** has the form

$$\mathbf{y} = \sum_{r=1}^{n} \mathbf{v}_r a_r \tag{28.18}$$

where the a_r are scalar *mode multipliers.* When **y** and the $\mathbf{v}_r$ are known, it is possible to evaluate the a_r by premultiplying both sides by $\mathbf{v}_s{}^T\mathbf{M}$. Because of the orthogonality relations of Eq. (28.15), all the terms on the right vanish except the one for which $r = s$. Inserting the value of the mode multiplier so obtained, the expansion can be rewritten as

$$\mathbf{y} = \sum_{r=1}^{n} \mathbf{v}_r \frac{\mathbf{v}_r{}^T\mathbf{M}\mathbf{y}}{\mathbf{v}_r{}^T\mathbf{M}\mathbf{v}_r} \tag{28.19}$$

or alternatively as

$$\mathbf{y} = \sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r{}^T \mathbf{M}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \mathbf{y} \tag{28.20}$$

The form of Eq. (28.19) emphasizes the decomposition into eigenvectors since the fraction on the right is just a scalar. The form of Eq. (28.20) is convenient when a large number of vectors **y** are to be decomposed, since the fractions on the right, which are now square matrices, must be computed only once. The form of Eq. (28.20) becomes more economical of computation time when more than n vectors **y** have to be expanded. A useful check on the calculation of the matrices on the right of Eq. (28.20) is provided by the identity

$$\sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r{}^T \mathbf{M}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} = \mathbf{I} \tag{28.21}$$

which follows from Eq. (28.20) because **y** is completely arbitrary.

An alternative expansion which is useful for expanding the excitation vector **f** is

$$\mathbf{f} = \sum_{r=1}^{n} \omega_r{}^2 \mathbf{M} \mathbf{v}_r a_r = \sum_{r=1}^{n} \mathbf{M} \mathbf{v}_r \frac{\mathbf{v}_r{}^T \mathbf{f}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \tag{28.22}$$

This may be viewed as an expansion of the excitation in terms of the *inertia force* amplitudes of the natural modes. The mode multiplier a_r has been evaluated by premultiplying by $\mathbf{v}_r{}^T$. A form analogous to Eq. (28.20) and an identity corresponding to Eq. (28.21) can easily be written.

RAYLEIGH'S QUOTIENT

If Eq. (28.11) is premultiplied by $\mathbf{v}^T$, the following scalar equation is obtained:

$$\mathbf{v}^T \mathbf{K} \mathbf{v} = \omega^2 \mathbf{v}^T \mathbf{M} \mathbf{v}$$

The positive definiteness of **M** guarantees that $\mathbf{v}^T \mathbf{M} \mathbf{v}$ is nonzero, so that it is permissible to solve for ω^2.

$$\omega^2 = \frac{\mathbf{v}^T \mathbf{K} \mathbf{v}}{\mathbf{v}^T \mathbf{M} \mathbf{v}} \tag{28.23}$$

This quotient is called "Rayleigh's quotient." It also may be derived by equating time averages of potential and kinetic energy under the assumption that the vibratory system is executing simple harmonic motion at frequency ω with amplitude ratios given by **v** or by equating the maximum value of kinetic energy to the maximum value of potential energy under the same assumption. Rayleigh's quotient has the following interesting properties.

1. When **v** is an eigenvector $\mathbf{v}_r$ of Eq. (28.11), then Rayleigh's quotient is equal to the corresponding eigenvalue $\omega_r{}^2$.
2. If **v** is an approximation to $\mathbf{v}_r$ with an error which is a *first-order* infinitesimal, then Rayleigh's quotient is an approximation to $\omega_r{}^2$ with an error which is a *second-order* infinitesimal; i.e., Rayleigh's quotient is *stationary* in the neighborhoods of the true eigenvectors.
3. As **v** varies through all of n-dimensional vector space, Rayleigh's quotient remains bounded between the smallest and largest eigenvalues.

A common engineering application of Rayleigh's quotient involves simply evaluating Eq. (28.23) for a trial vector $\mathbf{v}$ which is selected on the basis of physical insight. When eigenvectors are obtained by approximate methods, Rayleigh's quotient provides a means of improving the accuracy in the corresponding eigenvalue. If the elements of an approximate eigenvector whose largest element is unity are correct to k decimal places, then Rayleigh's quotient can be expected to be correct to about $2k$ significant decimal places.

Perturbation Formulas. The perturbation formulas which follow provide the basis for estimating the changes in the eigenvalues and the eigenvectors which result from *small* changes in the stiffness and inertia parameters of a system. The formulas are strictly accurate only for infinitesimal changes but are useful approximations for *small* changes. They may be used by the designer to estimate the effects of a proposed change in a vibratory system and may also be used to analyze the effects of minor errors in the measurement of the system properties. Iterative procedures for the solution of eigenvalue problems can be based on these formulas. They are employed here to obtain approximations to the complex eigenvalues and eigenvectors of a lightly damped vibratory system in terms of the corresponding solutions for the same system without damping.

Suppose that the modal matrix $\mathbf{V}$ and the spectral matrix $\mathbf{\Omega}^2$ for the eigenvalue problem

$$\mathbf{KV} = \mathbf{MV\Omega}^2 \tag{28.14}$$

are known. Consider the perturbed eigenvalue problem

$$\mathbf{K}_* \mathbf{V}_* = \mathbf{M}_* \mathbf{V}_* \mathbf{\Omega}_*^2$$

where

$$\mathbf{K}_* = \mathbf{K} + d\mathbf{K} \qquad \mathbf{M}_* = \mathbf{M} + d\mathbf{M}$$

$$\mathbf{V}_* = \mathbf{V} + d\mathbf{V} \qquad \mathbf{\Omega}_*^2 = \mathbf{\Omega}^2 + d\mathbf{\Omega}^2$$

The perturbation formula for the elements $d\omega_r^2$ of the diagonal matrix $d\mathbf{\Omega}^2$ is

$$d\omega_r^2 = \frac{\mathbf{v}_r^T \, d\mathbf{K} \, \mathbf{v}_r - \omega_r^2 \mathbf{v}_r^T \, d\mathbf{M} \, \mathbf{v}_r}{\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r} \tag{28.24}$$

Thus in order to determine the change in a single eigenvalue due to changes in $\mathbf{M}$ and $\mathbf{K}$, it is necessary to know only the corresponding unperturbed eigenvalue and eigenvector. To determine the change in a single eigenvector, however, it is necessary to know *all* the unperturbed eigenvalues and eigenvectors. The following algorithm may be used to evaluate the perturbations of both the modal matrix and the spectral matrix. Calculate

$$\mathbf{F} = \mathbf{V}^T \, d\mathbf{K} \, \mathbf{V} - \mathbf{V}^T \, d\mathbf{M} \, \mathbf{V\Omega}^2$$

and

$$\mathbf{L} = \mathbf{V}^T \mathbf{M} \mathbf{V}$$

The matrix $\mathbf{L}$ is a diagonal matrix of positive elements and hence is easily inverted. Continue calculating

$$\mathbf{G} = \mathbf{L}^{-1}\mathbf{F} = [g_{jk}] \qquad \text{and} \qquad \mathbf{H} = [h_{jk}]$$

where

$$h_{jk} = \begin{cases} 0 & \text{if } \omega_j^2 = \omega_k^2 \\ \dfrac{g_{jk}}{\omega_k^2 - \omega_j^2} & \text{if } \omega_j^2 \neq \omega_k^2 \end{cases}$$

Then, finally, the perturbations of the modal matrix and the spectral matrix are given by

$$d\mathbf{V} = \mathbf{VH} \qquad d\boldsymbol{\Omega}^2 = \left[g_{jj} \right] \tag{28.25}$$

These formulas are derived by taking the total differential of Eq. (28.14), premultiplying each term by $\mathbf{V}^T$, and using a relation derived by taking the transpose of Eq. (28.14). An interesting property of the perturbation approximation is that the change in each eigenvector is orthogonal with respect to $\mathbf{M}$ to the corresponding unperturbed eigenvector; i.e.,

$$\mathbf{v}_j^T \mathbf{M}\, d\mathbf{v}_j = 0$$

VIBRATIONS OF SYSTEMS WITHOUT DAMPING

In this section the damping matrix $\mathbf{C}$ is neglected in Eq. (28.8), leaving the general formulation in the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{28.26}$$

Solutions are outlined for the following three cases: free vibration ($\mathbf{f} = 0$), steady-state forced sinusoidal vibration ($\mathbf{f} = \Re\{\mathbf{d}e^{j\omega t}\}$, where $\mathbf{d}$ is a column vector of driving-force amplitudes), and the response to general excitation ($\mathbf{f}$ an arbitrary function of time). The first two cases are contained in the third, but for the sake of clarity each is described separately.

FREE VIBRATION WITH SPECIFIED INITIAL CONDITIONS

It is desired to find the solution $\mathbf{x}(t)$ of Eq. (28.26) when $\mathbf{f} = 0$ which satisfies the initial conditions

$$\mathbf{x} = \mathbf{x}(0) \qquad \dot{\mathbf{x}} = \dot{\mathbf{x}}(0) \tag{28.27}$$

at $t = 0$ where $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ are columns of prescribed initial displacements and velocities. The differential equation to be solved is identical with Eq. (28.10), which led to the matrix eigenvalue problem in the preceding section. Assuming that the solution of the eigenvalue problem is available, the general solution of the differential equation is given by an arbitrary superposition of the natural modes

$$\mathbf{x} = \sum_{r=1}^{n} \mathbf{v}_r (a_r \cos \omega_r t + b_r \sin \omega_r t)$$

where the $\mathbf{v}_r$ are the eigenvectors or natural modes, the ω_r are the natural frequencies, and the a_r and b_r are $2n$ constants of integration. The corresponding velocity is

$$\dot{\mathbf{x}} = \sum_{r=1}^{n} \mathbf{v}_r \omega_r (-a_r \sin \omega_r t + b_r \cos \omega_r t)$$

Setting $t = 0$ in these expressions and substituting in the initial conditions of Eq. (28.27) provides $2n$ simultaneous equations for determination of the constants of integration.

$$\sum_{r=1}^{n} \mathbf{v}_r a_r = \mathbf{x}(0) \qquad \sum_{r=1}^{n} \mathbf{v}_r \omega_r b_r = \dot{\mathbf{x}}(0)$$

These equations may be interpreted as eigenvector expansions of the initial displacement and velocity. The constants of integration can be evaluated by the same technique used to obtain the mode multipliers in Eq. (28.19). Using the form of Eq. (28.20), the solution of the free vibration problem then becomes

$$\mathbf{x}(t) = \sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r{}^T \mathbf{M}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \left\{ \mathbf{x}(0) \cos \omega_r t + \frac{1}{\omega_r} \dot{\mathbf{x}}(0) \sin \omega_r t \right\} \tag{28.28}$$

STEADY-STATE FORCED SINUSOIDAL VIBRATION

It is desired to find the steady-state solution to Eq. (28.26) for single-frequency sinusoidal excitation **f** of the form

$$\mathbf{f} = \Re \{ \mathbf{d} e^{j\omega t} \}$$

where **d** is a column vector of driving force amplitudes (these may be complex to permit differences in phase for the various components). The solution obtained is a useful approximation for lightly damped systems provided that the forcing frequency ω is not too close to a natural frequency ω_r. For resonance and near-resonance conditions it is necessary to include the damping as indicated in the section which follows the present discussion.

The steady-state solution desired is assumed to have the form

$$\mathbf{x} = \Re \{ \mathbf{a} e^{j\omega t} \}$$

where **a** is an unknown column vector of response amplitudes. When **f** and **x** are inserted in Eq. (28.26), the following set of simultaneous equations for the elements of **a** is obtained:

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{a} = \mathbf{d} \tag{28.29}$$

If ω is not a natural frequency, the square matrix $\mathbf{K} - \omega^2 \mathbf{M}$ is nonsingular and may be inverted to yield

$$\mathbf{a} = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{d}$$

as a complete solution for the response amplitudes in terms of the driving force amplitudes. This solution is useful if several force amplitude distributions are to be studied while the excitation frequency ω is held constant. The process requires repeated inversions if a range of frequencies is to be studied.

An alternative procedure which permits a more thorough study of the effect of frequency variation is available if the natural modes and frequencies are known. The driving-force vector **d** is represented by the eigenvector expansion of Eq. (28.22), and the response vector **a** is represented by the eigenvector expansion of Eq. (28.18):

$$\mathbf{d} = \sum_{r=1}^{n} \frac{\mathbf{M}\mathbf{v}_r\mathbf{v}_r^T}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r}\,\mathbf{d} \qquad \mathbf{a} = \sum_{r=1}^{n} \mathbf{v}_r c_r$$

where the c_r are unknown coefficients. Substituting these into Eq. (28.29), and making use of the fundamental eigenvalue relation of Eq. (28.11), leads to

$$\sum_{r=1}^{n} (\omega_r^2 - \omega^2)\mathbf{M}\mathbf{v}_r c_r = \sum_{r=1}^{n} \frac{\mathbf{M}\mathbf{v}_r\mathbf{v}_r^T}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r}\,\mathbf{d}$$

This equation can be uncoupled by premultiplying both sides by $\mathbf{v}_r^T$ and using the orthogonality condition of Eq. (28.15) to obtain

$$(\omega_r^2 - \omega^2)\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r c_r = \mathbf{v}_r^T\mathbf{d}$$

$$c_r = \frac{1}{\omega_r^2 - \omega^2}\,\frac{\mathbf{v}_r^T\mathbf{d}}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r}$$

The final solution is then assembled by inserting the c_r back into $\mathbf{a}$ and $\mathbf{a}$ back into $\mathbf{x}$.

$$\mathbf{x} = \Re\left\{\sum_{r=1}^{n} \frac{e^{j\omega t}}{\omega_r^2 - \omega^2}\,\frac{\mathbf{v}_r\mathbf{v}_r^T}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r}\,\mathbf{d}\right\} \tag{28.30}$$

This form clearly indicates the effect of frequency on the response.

RESPONSE TO GENERAL EXCITATION

It is now desired to obtain the solution to Eq. (28.26) for the general case in which the excitation $\mathbf{f}(t)$ is an arbitrary vector function of time and for which initial displacements $\mathbf{x}(0)$ and velocities $\dot{\mathbf{x}}(0)$ are prescribed. If the natural modes and frequencies of the system are available, it is again possible to split the problem up into n single degree-of-freedom response problems and to indicate a formal solution.

Following a procedure similar to that just used for steady-state forced sinusoidal vibrations, an eigenvector expansion of the solution is assumed:

$$\mathbf{x}(t) = \sum_{r=1}^{n} \mathbf{y}_r c_r(t)$$

where the c_r are unknown functions of time and the known excitation $\mathbf{f}(t)$ is expanded according to Eq. (28.22). Inserting these into Eq. (28.26) yields

$$\sum_{r=1}^{n} (\mathbf{M}\mathbf{v}_r\ddot{c}_r + \mathbf{K}\mathbf{v}_r c_r) = \sum_{r=1}^{n} \frac{\mathbf{M}\mathbf{v}_r\mathbf{v}_r^T}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r}\,\mathbf{f}(t)$$

Using Eq. (28.11) to eliminate $\mathbf{K}$ and premultiplying by $\mathbf{v}_r^T$ to uncouple the equation,

$$\ddot{c}_r + \omega_r^2 c_r^2 = \frac{\mathbf{v}_r^T\mathbf{f}(t)}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r} \tag{28.31}$$

is obtained as a single second-order differential equation for the time behavior of the rth mode multiplier. The initial conditions for c_r can be obtained by making eigenvector expansions of $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ as was done previously for the free vibration case. Formal solutions to Eq. (28.29) can be obtained by a number of methods, including Laplace transforms and variation of parameters. When these mode multipliers are substituted back to obtain $\mathbf{x}$, the general solution has the following appearance:

$$\mathbf{x}(t) = \sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r{}^T \mathbf{M}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \left\{ \mathbf{x}(0) \cos \omega_r t + \frac{1}{\omega_r} \dot{\mathbf{x}}(0) \sin \omega_r t \right\}$$

$$+ \sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r{}^T}{\omega_r \mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \int_0^t \mathbf{f}(t') \sin \{\omega_r (t - t')\} \, dt' \quad (28.32)$$

The integrals involving the excitation can be evaluated in closed form if the elements $f_j(t)$ of $\mathbf{f}(t)$ are simple (e.g., step functions, ramps, single sine pulses, etc.). When the $f_j(t)$ are more complicated, numerical results can be obtained by using integration software.